Discussion 0A

Logical Symbols

$\wedge$	and
$\vee$	or
$\neg$	not
$\exists$	there exists
$\forall$	for all
$\Rightarrow$	implies

Other Mathematical Symbols

$\mid$	divides
$\in$	in
$\subseteq$	subset
$\cup$	union
$\cap$	intersection
$\setminus$	relative complement

Implication

$P \Rightarrow Q$ "if P, then Q"

- equivalent to $Q \vee \neg P$
- To negate, use DeMorgan's: $\neg(P \Rightarrow Q) \equiv \neg((\neg P) \vee Q) \equiv P \wedge \neg Q$

1 Truth Tables

Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

(a) $P \wedge (Q \vee P) \;\equiv\; P \wedge Q$

(b) $(P \vee Q) \wedge R \equiv (P \wedge R) \vee (Q \wedge R)$

(c) $(P \wedge Q) \vee R \equiv (P \vee R) \wedge (Q \vee R)$

a)

P	Q	$P \wedge (Q \vee P)$	$P \wedge Q$
T	T	T	T
T	F	T	F
F	T	F	F
F	F	F	F

not equivalent

P	Q	R	$(P \vee Q) \wedge R$	$(P \wedge R) \vee (Q \wedge R)$	$(P \wedge Q) \vee R$	$(P \vee R) \wedge (Q \vee R)$
T	T	T	T	T	T	T
T	T	F	F	F	T	T
T	F	T	T	T	T	T
T	F	F	F	F	F	F
F	T	T	T	T	T	T
F	T	F	F	F	F	F
F	F	T	F	F	T	T
F	F	F	F	F	F	F

b) equivalent

c) equivalent

2 Propositional Practice

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

(a) There is a real number which is not rational.

(b) All integers are natural numbers or are negative, but not both.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

(d) $(\forall x \in \mathbb{Z})\ (x \in \mathbb{Q})$

(e) $(\forall x \in \mathbb{Z})\ (((2 \mid x) \vee (3 \mid x)) \implies (6 \mid x))$

(f) $(\forall x \in \mathbb{N})\ ((x > 7) \implies ((\exists a, b \in \mathbb{N})\ (a + b = x)))$

a) $(\exists x \in \mathbb{R})(x \notin \mathbb{Q})$ True, $x = \sqrt{2}$

b) $(\forall x \in \mathbb{Z})(((x \in \mathbb{N}) \vee (x < 0)) \wedge (\neg(x \in \mathbb{N} \wedge x < 0)))$ True, $x \geq 0$ if and only if $x \in \mathbb{N}$

c) $(\forall x \in \mathbb{N})((6 \mid x) \Rightarrow (2 \mid x) \vee (3 \mid x))$ True, if $x = 6k$ then $x = 2(3k)$

d) Every integer is rational. True, every $x \in \mathbb{Z}$ can be written $\frac{x}{1}$

e) If an integer is divisible by 2 or 3, it is divisible by 6. False, 2 is divisible by 2 but not 6

f) Every natural number greater than 7 is the sum of two natural numbers.

True, for any $x \in \mathbb{N}$ where $x > 7$, can write $x + 0 = x$.

3 Converse and Contrapositive

Consider the statement "if a natural number is divisible by 4, it is divisible by 2".

(a) Write the statement in propositional logic. Prove that it is true or give a counterexample.

(b) Write the inverse of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The inverse of an implication $P \implies Q$ is $\neg P \implies \neg Q$.)

(c) Write the converse of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

(d) Write the contrapositive of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

a) $(\forall x \in \mathbb{N})((4|x) \Rightarrow (2|x))$

True. If $x = 4k$ for $k \in \mathbb{N}$ then $x = 2(2k)$ where $2k \in \mathbb{N}$, so $2|x$.

b) $(\forall x \in \mathbb{N})((4 \nmid x) \Rightarrow (2 \nmid x))$

False. Consider $x = 2$. Then $4 \nmid x$ but $2|x$.

c) $(\forall x \in \mathbb{N})((2|x) \Rightarrow (4|x))$ False. Consider $x = 2$.

d) $(\forall x \in \mathbb{N})((4 \nmid x) \Rightarrow (2 \nmid x))$ True, b/c original was true

4 Equivalences with Quantifiers

Evaluate whether the expressions on the left and right sides are equivalent in each part, and briefly justify your answers.

(a)	$\forall x\ ((\exists y\ Q(x,y)) \Rightarrow P(x))$	$\forall x\ \exists y\ (Q(x,y) \Rightarrow P(x))$
(b)	$\neg \exists x\ \forall y\ (P(x,y) \Rightarrow \neg Q(x,y))$	$\forall x\ ((\exists y\ P(x,y)) \land (\exists y\ Q(x,y)))$
(c)	$\forall x\ \exists y\ (P(x) \Rightarrow Q(x,y))$	$\forall x\ (P(x) \Rightarrow (\exists y\ Q(x,y)))$

a) Not equivalent. Let $Q(x,y)$ be "$x > y$" and $P(x)$ be "$x > 5$". Then $\forall x((\exists y\ Q(x,y)) \Rightarrow P(x))$ is saying "for every number x, if there exists a smaller number y then $x > 5$" which is false. But $\forall x \exists y (Q(x,y) \Rightarrow P(x))$ is saying "for all x, there exists a number y such that if $x > y$, then $x > 5$" which is true.

b) Not equivalent. Using DeMorgan's Law on $\neg \exists x \forall y (P(x,y) \Rightarrow \neg Q(x,y))$, we get $\forall x \exists y (P(x,y) \land Q(x,y))$ which is not equivalent to $\forall x (\exists y\ P(x,y)) \land (\exists y\ Q(x,y))$

c) Equivalent. Rewrite $P(x) \Rightarrow Q(x,y)$ as $\neg P(x) \lor Q(x,y)$ and separate into cases. If $P(x)$ is false then both sides are same; if

$P(x)$ is true then both sides are still equivalent.