1 Contraposition

Prove the statement "if $a+b < c+d$, then $a < c$ or $b < d$".

Contrapositive: If $a \geq c$ and $b \geq d$, then $a+b \geq c+d$.

Proof:

$$\begin{array}{rl} & a \geq c \\ + & b \geq d \\ \hline & a+b \geq c+d \end{array}$$

Add inequalities together.

2 Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice.
(Hint: The Pigeonhole Principle states that if n items are placed in m containers, where $n > m$, at least one container must contain more than one item. You may use this without proof.)

Suppose all n people have a different number of friends. The possible number of friends a person has is in $\{0, 1, 2, \ldots, n-1\}$. Since there are n possible "buckets", every bucket needs to have 1 person in it. However, this would mean there is 1 person with 0 friends and 1 person with $n-1$ friends. This is a contradiction, because having $n-1$ friends would mean being friends with everyone else, including the person with 0 friends. Thus one of 0 or $n-1$ must be empty, so there must be at least two people who share the same number of friends.

3 Pebbles

Suppose you have a rectangular array of pebbles, where each pebble is either red or blue. Suppose that for every way of choosing one pebble from each column, there exists a red pebble among the chosen ones. Prove that there must exist an all-red column.

Contrapositive: If there is no all-red column, then it is possible to pick a blue pebble from each column.

Proof: Suppose there is no all-red column. That means there exists a blue pebble in each column. From each column, we can pick a blue pebble. Thus, we have selected a pebble from every column such that none are red.

4 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets.

Recall: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(b) $f(A \cup B) = f(A) \cup f(B)$.

a) Claim 1: $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$

Proof: Suppose $x \in f^{-1}(A \cup B)$.

Then $f(x) \in A \cup B$, so $f(x) \in A$ or $f(x) \in B$.

This means $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$, which means $x \in f^{-1}(A) \cup f^{-1}(B)$.

Claim 2: $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$

Proof: $x \in f^{-1}(A) \cup f^{-1}(B) \Rightarrow x \in f^{-1}(A)$ or $x \in f^{-1}(B)$

$\Rightarrow f(x) \in A$ or $f(x) \in B$

$\Rightarrow f(x) \in A \cup B$

$\Rightarrow x \in f^{-1}(A \cup B)$

Since $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$, this means $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. $\square$

b) Claim 1: $f(A \cup B) \subseteq f(A) \cup f(B)$

Proof: $y \in f(A \cup B) \Rightarrow y = f(x)$ for some $x \in A \cup B$

$\Rightarrow y = f(x)$ where $x \in A$ or

$y = f(x)$ where $x \in B$

$\Rightarrow y \in f(A)$ or $y \in f(B)$

$\Rightarrow y \in f(A) \cup f(B)$

Claim 2: $f(A) \cup f(B) \subseteq f(A \cup B)$

Proof: $y \in f(A) \cup f(B) \Rightarrow y \in f(A)$ or $y \in f(B)$

$\Rightarrow y = f(x)$ for some $x \in A$ or

$y = f(x)$ for some $x \in B$

$\Rightarrow y = f(x)$ for some $x \in A \cup B$

$\Rightarrow y \in f(A \cup B)$

Claim 1 and Claim 2 together imply $f(A \cup B) = f(A) \cup f(B)$. $\square$