$$E[f(X)] = \sum_x f(x) P[X=x]$$

Linearity of Expectation: $E[X+Y] = E[X] + E[Y]$

$$E[cX] = cE[X]$$

1 Beast Arcade

One day you find yourself inside the Mr. Beast Arcade, which is full of games that pay YOU to play them!

(a) In the first game, Chandler hands you a crisp $20 bill up front. Then, he flips a coin that shows heads with probability p repeatedly, stopping when a heads comes up for the first time. You receive an additional dollar for each flip. How much money will you get in expectation?

$$X = \text{number of flips} \qquad X \sim \text{Geometric}(p)$$

$$E[20 + X] = 20 + E[X] = 20 + \frac{1}{p}$$

(b) In the next game, Karl rolls a fair 6-sided die. He then calculates 2^x, where x is the result of that die and hands you that much money. What is the expected amount of money you'll receive?

$$E[2^x] = \frac{1}{6}\left(2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6\right)$$

$$= 21$$

(c) For the last game, Jimmy makes your friend flip a fair coin $10,000$ times in a row, keeping track of the number of heads that show up. He then hands you a briefcase filled with $1,000$ and says he will also pay you $5 for each head that comes up. Let X be a random variable representing the number of heads your friend flips. Use it to come up with an expression for Y, a random variable representing the total amount of money you'll receive.

$$Y = 1000 + 5X$$

(d) What is $E[Y]$? What about $P[Y = 26,000]$?

$$E[Y] = 1000 + 5E[x] = 26000$$

$$P[Y = 26000] = P[X = 5000] = \binom{10000}{5000}\left(\frac{1}{2}\right)^{10000}$$

2 Linearity

Solve each of the following problems using linearity of expectation. Explain your methods clearly.

(a) In an arcade, you play game A 10 times and game B 20 times. Each time you play game A, you win with probability $1/3$ (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game B is similar, but you win with probability $1/5$, and if you win you get 4 tickets. What is the expected total number of tickets you receive?

$$A_i = \text{win } i^{th} \text{ game of } A$$
$$B_j = \text{win } j^{th} \text{ game of } B$$

$$E[A_i] = \frac{1}{3}$$
$$E[B_j] = \frac{1}{5}$$

$$E[3A_1 + \ldots + 3A_{10} + 4B_1 + \ldots + 4B_{20}] = 3 \cdot 10 \cdot \frac{1}{3} + 4 \cdot 20 \cdot \frac{1}{5} = 26$$

(b) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence "book" appears? (*Hint*: Consider where the sequence "book" can appear in the string.)

$$X = X_1 + \ldots \times X_{999997}$$

$$E[X] = E[X_1 + \ldots + X_{999997}]$$
$$= E[X_1] + \ldots + E[X_{999997}]$$
$$= 999997 \left(\frac{1}{26}\right)^4 \approx 2.19$$

3 Family Planning

Mr. and Mrs. Brown decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let G denote the numbers of girls that the Browns have. Let C be the total number of children they have.

(a) Determine the sample space, along with the probability of each sample point.

$$\Omega = \{g, \ bg, \ bbg, \ bbb\}$$
$$\tfrac{1}{2} \quad \tfrac{1}{4} \quad \tfrac{1}{8} \quad \tfrac{1}{8}$$

(b) Compute the joint distribution of G and C. Fill in the table below.

	$C = 1$	$C = 2$	$C = 3$
$G = 0$	0	0	$\tfrac{1}{8}$
$G = 1$	$\tfrac{1}{2}$	$\tfrac{1}{4}$	$\tfrac{1}{8}$

(c) Use the joint distribution to compute the marginal distributions of G and C and confirm that the values are as you'd expect. Fill in the tables below.

$\mathbb{P}(G = 0)$	$\tfrac{1}{8}$
$\mathbb{P}(G = 1)$	$\tfrac{7}{8}$

$\mathbb{P}(C = 1)$	$\mathbb{P}(C = 2)$	$\mathbb{P}(C = 3)$
$\tfrac{1}{2}$	$\tfrac{1}{4}$	$\tfrac{1}{4}$

(d) Are G and C independent?

No. $P[G = 0, \ C = 1] \neq P[G = 0] \, P[C = 1]$

(e) What is the expected number of girls the Browns will have? What is the expected number of children that the Browns will have?

$$E[G] = \tfrac{7}{8}$$

$$E[C] = \tfrac{1}{2} \cdot 1 + \tfrac{1}{4} \cdot 2 + \tfrac{1}{4} \cdot 3$$
$$= \tfrac{7}{4}$$