Poisson Distribution

$X \sim Pois(\lambda)$

$P[X=k] = \frac{\lambda^k}{k!} e^{-\lambda}$ $\qquad$ $E[X] = \lambda$ $\qquad$ $Var(X) = \lambda$

Notice that $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = 1.$

Variance

- measures how "spread out" a random variable
- expected squared distance from mean

$$Var(X) = E[(X-\mu)^2] = E[X^2] - E[X]^2$$

Properties of Variance:

$Var(cX) = c^2 Var(X)$

$Var(X+Y) = Var(X) + Var(Y)$ if X and Y are independent

Covariance

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - E[X]E[Y]$$

Properties of Covariance:

1. $Cov(X,X) = Var(X)$
2. If X and Y are independent, then $Cov(X,Y) = 0$.
3. $Cov(aX+bY, Z) = a\,Cov(X,Z) + b\,Cov(Y,Z)$
4. $Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

Common Discrete Random Variable Distributions

Distribution	Possible Values	$P[X=k]$	$E[X]$	$Var(X)$
Uniform$\{a,\ldots,b\}$	$\{a, a+1, \ldots, b\}$	$\frac{1}{b-a+1}$	$\frac{a+b}{2}$	$\frac{(b-a+1)^2-1}{12}$
Bernoulli(p)	$\{0, 1\}$	$\begin{cases} 1-p & \text{if } k=0 \\ p & \text{if } k=1 \end{cases}$	p	$p(1-p)$
Binomial(n, p)	$\{0, 1, \ldots, n\}$	$\binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$
Geometric(p)	$\{1, 2, \ldots\}$	$(1-p)^{k-1} p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson(λ)	$\{0, 1, 2, \ldots\}$	$\frac{\lambda^k}{k!} e^{-\lambda}$	λ	λ
Hypergeometric(N, B, n)	$\{0, \ldots, \min(n, B)\}$	$\frac{\binom{B}{k}\binom{N-B}{n-k}}{\binom{N}{n}}$	$n\frac{B}{N}$	$n \frac{B}{N} \frac{N-B}{N} \frac{N-n}{N-1}$

1 Sum of Poisson Variables

Assume that you were given two independent Poisson random variables X_1, X_2. Assume that the first has mean λ_1 and the second has mean λ_2. Prove that $X_1 + X_2$ is a Poisson random variable with mean $\lambda_1 + \lambda_2$.

Hint: Recall the binomial theorem.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$X_1 \sim \text{Poisson}(\lambda_1)$, $X_2 \sim \text{Poisson}(\lambda_2)$

$Y = X_1 + X_2$

$$P[Y=n] = \sum_{k=0}^{n} P[X_1 = k, X_2 = n-k]$$

$$= \sum_{k=0}^{n} \frac{\lambda_1^k}{k!} e^{-\lambda_1} \frac{\lambda_2^{n-k}}{(n-k)!} e^{-\lambda_2}$$

$$= \frac{e^{-\lambda_1} e^{-\lambda_2}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \lambda_1^k \lambda_2^{n-k} = \frac{e^{-\lambda_1-\lambda_2}}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k} = \frac{e^{-\lambda_1-\lambda_2}}{n!} (\lambda_1+\lambda_2)^n$$

$Y \sim \text{Poisson}(\lambda_1+\lambda_2)$

2 Variance

(a) Let X be a random variable representing the outcome of the roll of one fair 6-sided die. What is $\text{Var}(X)$?

$\text{Var}(X) = E[X^2] - E[X]^2$

$E[X] = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$

$$E[X^2] = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2)$$

$$= \frac{91}{6}$$

$$\text{Var}(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

(b) Let Z be a random variable representing the average of n rolls of a fair die 6-sided die. What is $\text{Var}(Z)$?

$$Z = \frac{1}{n}(X_1 + \ldots + X_n)$$

$$\text{Var}(Z) = \frac{1}{n^2}(\text{Var}(X_1) + \ldots + \text{Var}(X_n))$$

$$= \frac{1}{n^2}\left(\frac{35}{12} + \ldots + \frac{35}{12}\right)$$

$$= \frac{1}{n^2}\left(\frac{35}{12} n\right)$$

$$= \frac{35}{12n}$$

3 Covariance

(a) We have a bag of 5 red and 5 blue balls. We take two balls uniformly at random from the bag without replacement. Let X_1 and X_2 be indicator random variables for the events of the first and second ball being red, respectively. What is $\text{cov}(X_1, X_2)$? Recall that $\text{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$.

$$E[X_1] = P[X_1 = 1] = \frac{1}{2}$$

$$E[X_2] = P[X_2 = 1] = \frac{1}{2} \cdot \frac{4}{9} + \frac{1}{2} \cdot \frac{5}{9} = \frac{1}{2}$$

$$E[X_1 X_2] = P[X_1 = 1, X_2 = 1] = \frac{1}{2} \cdot \frac{4}{9} = \frac{2}{9}$$

$$\begin{aligned} \text{Cov}(X_1, X_2) &= E[X_1 X_2] - E[X_1]\,E[X_2] \\ &= \frac{2}{9} - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ &= -\frac{1}{36} \end{aligned}$$

(b) Now, we have two bags A and B, with 5 red and 5 blue balls each. Draw a ball uniformly at random from A, record its color, and then place it in B. Then draw a ball uniformly at random from B and record its color. Let X_1 and X_2 be indicator random variables for the events of the first and second draws being red, respectively. What is $\text{cov}(X_1, X_2)$?

$$E[X_1] = \frac{1}{2}$$

$$E[X_2] = \frac{1}{2} \cdot \frac{6}{11} + \frac{1}{2} \cdot \frac{5}{11} = \frac{1}{2}$$

$$E[X_1 X_2] = P[X_1 = 1, X_2 = 1] = \frac{1}{2} \cdot \frac{6}{11} = \frac{3}{11}$$

$$\text{Cov}(X_1, X_2) = \frac{3}{11} - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{44}$$