# Continuous Distributions

probability density function (PDF) : a function $f(x)$ representing the "probability per unit length" of a continuous random variable $X$

$$P[a \le X \le b] = \int_a^b f(x)\,dx \qquad \text{for all } a \le b$$

A valid PDF satisfies:

1. $f(x) \ge 0$ for all $x \in \mathbb{R}$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

cumulative distribution function (CDF) = $F(x) = P[X \le x]$

$$F(x) = \int_{-\infty}^{x} f(z)\,dz$$

$$f(x) = \frac{d}{dx} F(x)$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$\begin{aligned} Var(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ &= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\int_{-\infty}^{\infty} x f(x)\,dx\right)^2 \end{aligned}$$

# Joint Distributions

Joint density function: $f(x, y)$

$$P[a \le X \le b,\ c \le Y \le d] = \int_c^d \int_a^b f(x, y)\,dx\,dy \qquad \text{for all } a \le b \text{ and } c \le d$$

1. $f(x, y) \ge 0$ for all $x, y \in \mathbb{R}$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\,dx\,dy = 1$

CS 70 Discrete Mathematics and Probability Theory
Spring 2022 Satish Rao and Koushik Sen **DIS 13A**

# 1 Continuous Intro

(a) Is

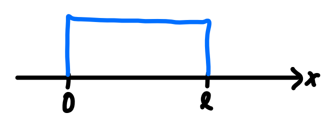
$$f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

a valid density function? Why or why not? Is it a valid CDF? Why or why not?

Yes, valid PDF: $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 2x\,dx = x^2 \Big|_0^1 = 1$ Not a valid CDF.

(b) Calculate $\mathbb{E}[X]$ and $\text{Var}(X)$ for $X$ with the density function

$$f(x) = \begin{cases} \frac{1}{\ell}, & 0 \le x \le \ell, \\ 0, & \text{otherwise.} \end{cases}$$




$\mathbb{E}[X] = \int_0^{\ell} \frac{x}{\ell}\,dx = \frac{1}{2\ell} x^2 \Big|_0^{\ell} = \frac{\ell}{2}$

$\mathbb{E}[X^2] = \int_0^{\ell} \frac{x^2}{\ell}\,dx = \frac{1}{3\ell} x^3 \Big|_0^{\ell} = \frac{\ell^2}{3}$

$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$
$= \frac{\ell^2}{3} - \left(\frac{\ell}{2}\right)^2 = \frac{\ell^2}{12}$

(c) Suppose $X$ and $Y$ are independent and have densities

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$f_Y(y) = \begin{cases} 1, & 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is their joint distribution? (Hint: for this part and the next, we can use independence in much the same way that we did in discrete probability)

$f_{X,Y}(x, y) = f_X(x)\, f_Y(y)$

$= \begin{cases} 2x & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

(d) Calculate $\mathbb{E}[XY]$ for the above $X$ and $Y$.

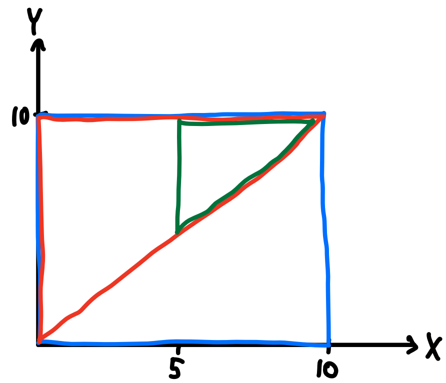
$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f(x, y)\,dx\,dy$

$= \int_0^1 \int_0^1 2x^2 y\,dx\,dy$

$= \int_0^1 \left(\frac{2}{3} x^3 y \Big|_{x=0}^{x=1}\right) dy = \int_0^1 \frac{2}{3} y\,dy = \frac{1}{3} y^2 \Big|_{y=0}^{y=1} = \frac{1}{3}$

## 2 Uniform Distribution

You have two fidget spinners, each having a circumference of 10. You mark one point on each spinner as a needle and place each of them at the center of a circle with values in the range $[0,10)$ marked on the circumference. If you spin both (independently) and let $X$ be the position of the first spinner's mark and $Y$ be the position of the second spinner's mark, what is the probability that $X \geq 5$, given that $Y \geq X$?




$$P[X \geq 5 \mid Y \geq X] = \frac{P[X \geq 5 \cap Y \geq X]}{P[Y \geq X]}$$

$$= \frac{\frac{1}{8}}{\frac{1}{2}}$$

$$= \frac{1}{4}$$

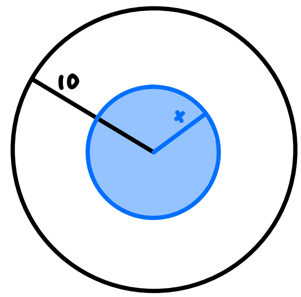
## 3 Darts Again

Edward and Khalil are playing darts on a circular dartboard.

Edward's throws are uniformly distributed over the entire dartboard, which has a radius of 10 inches. Khalil has good aim; the distance of his throws from the center of the dartboard follows an exponential distribution with parameter $\frac{1}{2}$.

Say that Edward and Khalil both throw one dart at the dartboard. Let $X$ be the distance of Edward's dart from the center, and $Y$ be the distance of Khalil's dart from the center of the dartboard. What is $\mathbb{P}[X < Y]$, the probability that Edward's throw is closer to the center of the board than Khalil's? Leave your answer in terms of an unevaluated integral.

[*Hint:* $X$ is not uniform over $[0,10]$. Solve for the distribution of $X$ by first computing the CDF of $X$, $\mathbb{P}[X < x]$.]




$$P[X \le x] = \frac{\text{Area of } \bigcirc}{\text{Area of dartboard}} = \frac{\pi x^2}{\pi (10)^2} = \frac{x^2}{100}$$

$$f_X(x) = \frac{d}{dx}\left(\frac{x^2}{100}\right) = \frac{1}{50}x$$

$$P[X < Y] = \int_0^{10} P[X < Y \mid X = x]\, f_X(x)\, dx$$

$$= \int_0^{10} P[Y > x]\, f_X(x)\, dx$$

$$= \int_0^{10} e^{-0.5x} \frac{x}{50}\, dx$$

$$= \int_0^{10} \frac{1}{50} x e^{-0.5x}\, dx \approx 0.0767$$