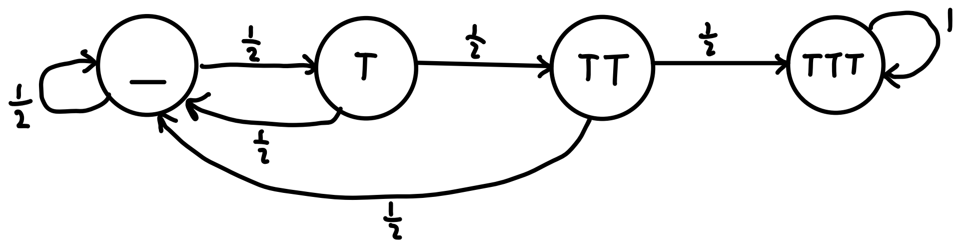
1 Three Tails

You flip a fair coin until you see three tails in a row. What is the average number of heads that you'll see until getting TTT?

Hint: How is this different than the number of *coins* flipped until getting TTT?



β_i = expected number of heads until getting TTT

$\beta_0 = \frac{1}{2}(1+\beta_0) + \frac{1}{2}\beta_1$

$\beta_1 = \frac{1}{2}(1+\beta_0) + \frac{1}{2}\beta_2$

$\beta_2 = \frac{1}{2}(1+\beta_0) + \frac{1}{2}\beta_3$

$\beta_3 = 0$

$\Rightarrow$

$\beta_2 = \frac{1}{2} + \frac{1}{2}\beta_0$

$\beta_1 = \frac{1}{2} + \frac{1}{2}\beta_0 + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\beta_0\right) = \frac{3}{4} + \frac{3}{4}\beta_0$

$\beta_0 = \frac{1}{2}(1+\beta_0) + \frac{1}{2}\left(\frac{3}{4} + \frac{3}{4}\beta_0\right)$

$= \frac{7}{8} + \frac{7}{8}\beta_0$

$\frac{1}{8}\beta_0 = \frac{7}{8}$

$\boxed{\beta_0 = 7}$

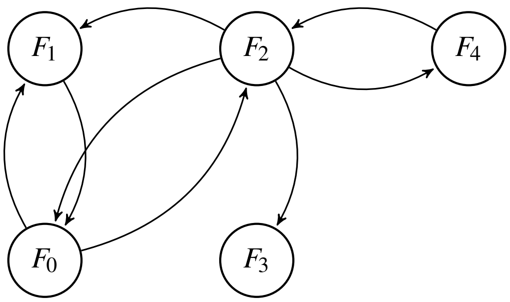
$\beta_1 = 6$

$\beta_2 = 4$

2 The Dwinelle Labyrinth

You have decided to take a humanities class this semester, a French class to be specific. Instead of a final exam, your professor has issued a final paper. You must turn in this paper *before* noon to the professor's office on floor 3 in Dwinelle, and it's currently 11:48 a.m.

Let Dwinelle be modeled by the following Markov chain. Instead of rushing to turn it in, we will spend valuable time computing whether or not we *could have* made it. Suppose walking between floors takes 1 minute.



$\beta_i = E[T_i]$

(a) Will you make it in time if you choose a floor to transition to uniformly at random? (If T_i is the number of steps needed to get to F_3 starting from F_i, where $i \in \{0,1,2,3,4\}$, is $\mathbb{E}[T_0] < 12$?)

$\beta_0 = 1 + \frac{1}{2}\beta_1 + \frac{1}{2}\beta_2$

$\beta_1 = 1 + \beta_0$

$\beta_2 = 1 + \frac{1}{4}\beta_0 + \frac{1}{4}\beta_1 + \frac{1}{4}\beta_3 + \frac{1}{4}\beta_4$

$\beta_3 = 0$

$\beta_4 = 1 + \beta_2$

$\beta_1 = 2 + \frac{1}{2}\beta_1 + \frac{1}{2}\beta_2$

$\frac{1}{2}\beta_1 = 2 + \frac{1}{2}\beta_2$

$\beta_1 = 4 + \beta_2$

$\beta_2 = 1 + \frac{1}{4}\left(1 + \frac{1}{2}(4+\beta_2) + \frac{1}{2}\beta_2\right) + \frac{1}{4}(4+\beta_2) + \frac{1}{4}(1+\beta_2)$

$= 3 + \frac{3}{4}\beta_2$

$\beta_2 = 12$ $\quad \boxed{\beta_0 = 15}$

$\beta_1 = 16$ $\quad \beta_4 = 13$

(b) Would you expect to make it in time, if for every floor, you order all accessible floors and are twice as likely to take higher floors? (If you are considering 1, 2, or 3, you will take each with probabilities $1/7$, $2/7$, $4/7$, respectively.)

$\beta_0 = 1 + \frac{1}{3}\beta_1 + \frac{2}{3}\beta_2$

$\beta_1 = 1 + \beta_0$

$\beta_2 = 1 + \frac{1}{15}\beta_0 + \frac{2}{15}\beta_1 + \frac{4}{15}\beta_3 + \frac{8}{15}\beta_4$

$\beta_3 = 0$

$\beta_4 = 1 + \beta_2$

$\beta_1 = 2 + \frac{1}{3}\beta_1 + \frac{2}{3}\beta_2$

$\beta_1 = 3 + \beta_2$

$\beta_0 = 1 + 1 + \frac{1}{3}\beta_2 + \frac{2}{3}\beta_2 = 2 + \beta_2$

$\beta_2 = 1 + \frac{1}{15}(2+\beta_2) + \frac{2}{15}(3+\beta_2) + \frac{8}{15}(1+\beta_2)$

$= \frac{31}{15} + \frac{11}{15}\beta_2$

$\frac{4}{15}\beta_2 = \frac{31}{15}$

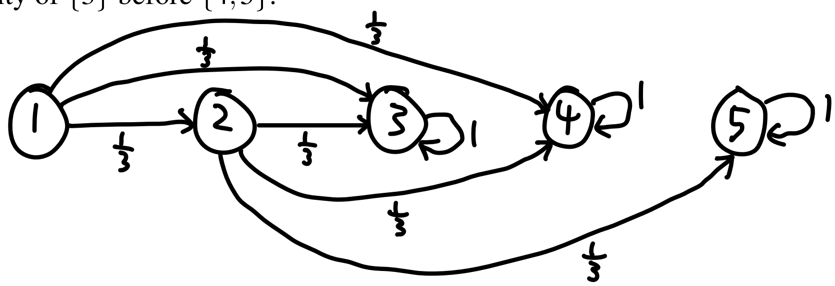
$\beta_2 = \frac{31}{4}$ $\quad \beta_0 = \frac{39}{4}$

$\beta_1 = \frac{43}{4}$ $\quad \beta_4 = \frac{35}{4}$

$\beta_0 = \frac{39}{4} = 9.75 < 12$

3 Skipping Stones

We consider a simple Markov chain model for skipping stones on a river, but with a twist: instead of trying to make the stone travel as far as possible, you want the stone to hit a target. Let the set of states be $\mathscr{X} = \{1,2,3,4,5\}$. State 3 represents the target, while states 4 and 5 indicate that you have overshot your target. Assume that from states 1 and 2, the stone is equally likely to skip forward one, two, or three steps forward. If the stone starts from state 1, compute the probability of reaching our target before overshooting, i.e. the probability of $\{3\}$ before $\{4,5\}$.



α_i = probability of reaching 3 before $\{4,5\}$, starting from state i

$$\alpha_1 = \frac{1}{3}\alpha_2 + \frac{1}{3}\alpha_3 + \frac{1}{3}\alpha_4$$

$$\alpha_2 = \frac{1}{3}\alpha_3 + \frac{1}{3}\alpha_4 + \frac{1}{3}\alpha_5$$

$$\alpha_3 = 1$$

$$\alpha_4 = 0$$

$$\alpha_5 = 0$$

$$\boxed{\alpha_1 = \frac{4}{9}}$$

$$\alpha_2 = \frac{1}{3}$$