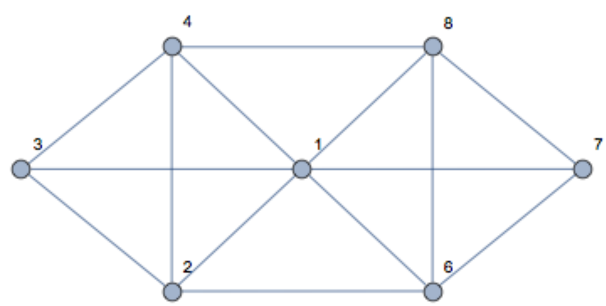
CS 70 Spring 2022 — Discrete Mathematics and Probability Theory — Koushik Sen and Satish Rao — DIS 2A

1 Optimal Candidates

In the notes, we proved that the propose-and-reject algorithm always outputs the job-optimal pairing. However, we never explicitly showed why it is guaranteed that putting every job with its optimal candidate results in a pairing at all. Prove by contradiction that no two jobs can have the same optimal candidate. (Note: your proof should not rely on the fact that the propose-and-reject algorithm outputs the job-optimal pairing.)

Suppose jobs J_1 and J_2 have the same optimal candidate C where $(J_1, C) \in M_1$ and $(J_2, C) \in M_2$. WLOG assume C prefers J_1 to J_2. Then (J_1, C) is a rogue couple in M_2 because J_1 must prefer C to whoever J_1 is paired with in M_2.

2 Eulerian Tour and Eulerian Walk



(a) Is there an Eulerian tour in the graph above? If no, give justification. If yes, provide an example.

No, vertices 3 and 7 have odd degree.

(b) Is there an Eulerian walk in the graph above? An Eulerian walk is a walk that uses each edge exactly once. If no, give justification. If yes, provide an example.

Yes.

3, 2, 4, 3, 1, 2, 6, 8, 4, 1, 7, 6, 8, 7

(c) What is the condition that there is an Eulerian walk in an undirected graph? Briefly justify your answer.

There are 0 or 2 vertices of odd degree, and the graph is connected (except isolated vertices).

[See official solutions for proof.]

3 Not everything is normal: Odd-Degree Vertices

Claim: Let $G = (V, E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

(i) Direct proof (e.g., counting the number of edges in G). *Hint: in lecture, we proved that* $\sum_{v \in V} \deg v = 2|E|$.

$$\sum_{v \in V} \deg v = 2|E|$$

V_o = odd degree vertices

V_e = even degree vertices

$$\sum_{v \in V_o} \deg v + \sum_{v \in V_e} \deg v = 2|E|$$

even + even = even

Thus $|V_o|$ must be even.

(ii) Induction on $m = |E|$ (number of edges)

Base Case: $m = 0$. No edges, so all vertices have degree 0.

Induction Hypothesis: Every graph with m edges has an even number of odd-degree vertices.

Inductive Step: Let G be a graph with $m+1$ edges.

Remove one edge (u,v) to create G'.

By inductive hypothesis, $|V_o(G')|$ is even.

Now add edge (u, v) back in.

$$|V_o(G)| = \begin{cases} |V_o(G')| - 2 & \text{if } u, v \in V_o(G') \\ |V_o(G')| & \text{if } u \in V_o(G'),\ v \notin V_o(G') \\ |V_o(G')| & \text{if } u \notin V_o(G'),\ v \in V_o(G') \\ |V_o(G')| + 2 & \text{if } u, v \notin V_o(G') \end{cases}$$

In all cases, $|V_o(G)|$ is even.

(iii) Induction on $n = |V|$ (number of vertices)

Skipped, see official solutions.

4 Coloring Trees

Prove that all trees with at least 2 vertices are *bipartite*: the vertices can be partitioned into two groups so that every edge goes between the two groups.

[*Hint:* Use induction on the number of vertices.]

Let $n = |V|$.

Base Case: $n = 2$

$L = \{u\}$, $R = \{v\}$

Induction Hypothesis: Every tree with k vertices is bipartite.

Inductive Step: Let T be a tree with $k+1$ vertices.

Remove a leaf node u and its incident edge (u, v).

By IH, the resulting tree is bipartite.

Now add node u back in. If $v \in L$, then put u in R. If $v \in R$, then put u in L.

Thus T is also bipartite.