sample k objects
out of n possibilities

| | Permutations (order matters) | Combinations (order doesn't matter) |
|---|---|---|
| with repeats | $n^k$ | $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ |
| without repeats | $\frac{n!}{(n-k)!}$ | $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ "n choose k" |

rearrange letters of ABCDE: $5 \times 4 \times 3 \times 2 \times 1 = 5!$

arranging 5 letters in a row: $26 \times 26 \times 26 \times 26 \times 26 = 26^5$

5 letters, no repeats: $26 \times 25 \times 24 \times 23 \times 22 = \frac{26!}{21!}$

# 1 Strings

What is the number of strings you can construct given:

(a) $n$ ones, and $m$ zeroes?

$$\frac{(n+m)!}{n!\,m!} = \binom{n+m}{n} = \binom{n+m}{m}$$

(b) $n_1$ A's, $n_2$ B's and $n_3$ C's?

$$\frac{(n_1+n_2+n_3)!}{n_1!\,n_2!\,n_3!}$$

(c) $n_1, n_2, \dots, n_k$ respectively of $k$ different letters?

$$\frac{(n_1+n_2+\dots+n_k)!}{n_1!\,n_2!\cdots n_k!}$$

## 2 You'll Never Count Alone

(a) An anagram of LIVERPOOL is any re-ordering of the letters of LIVERPOOL, i.e., any string made up of the letters L, I, V, E, R, P, O, O, L in any order. For example, IVLERPOOL and POLIVOLRE are anagrams of LIVERPOOL but PIVEOLR and CHELSEA are not. The anagram does not have to be an English word.

How many different anagrams of LIVERPOOL are there?

$$\frac{9!}{2!\,2!}$$

(b) How many solutions does $y_0 + y_1 + \cdots + y_k = n$ have, if each $y$ must be a non-negative integer?

$\star\ \star \mid \star\ \star\ \star \mid\mid \star\ \star$ n stars

$y_0 \quad y_1 \quad y_2 \quad y_3$ k bars

$$\frac{(n+k)!}{n!\,k!} = \binom{n+k}{k} = \binom{n+k}{n}$$

(c) How many solutions does $y_0 + y_1 + \cdots + y_k = n$ have, if each $y$ must be a positive integer?

Let $x_i = y_i - 1$.

$$x_0 + x_1 + \ldots + x_k = n - k - 1$$

$$\frac{(n-1)!}{(n-k-1)!\,k!} = \binom{n-1}{k}$$

## 3 The Count

(a) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

★ ★ | | ★ | | | | ★ | ★ ★ | | ★

9 8 7 6 5 4 3 2 1 0

7 stars

9 bars

$$\frac{16!}{9!\,7!} = \binom{16}{9}$$

(b) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he have now?

Choose 7 out of the 10 digits:

$\underline{9}$ 8 $\underline{7}$ 6 $\underline{5}$ 4 $\underline{3}$ $\underline{2}$ $\underline{1}$ $\underline{0}$

$$\binom{10}{7}$$

(c) The Count now wants to make a password to secure his phone. His password must be exactly 10 digits long and can only contain the digits 0 and 1. On top of that, he also wants it to contain at least five consecutive 0's. How many possible passwords can he make?

00000 _ _ _ _ _ $2^5$
100000 _ _ _ _ $2^4$
_ 100000 _ _ _ $2^4$
_ _ 100000 _ _ $2^4$
_ _ _ 100000 _ $2^4$
_ _ _ _ 100000 $2^4$

$2^5 + 5 \cdot 2^4 = 112$