CS 70 Discrete Mathematics and Probability Theory
Spring 2022 Koushik Sen and Satish Rao **DIS 6A**

1 Farmer's Market

Suppose you want k items from the farmer's market. Count how many ways you can do this, assuming:

(a) There are pumpkins and apples at the market.

$k-1$

(b) There are pumpkins, apples, oranges, and pears at the market.

$\binom{k+3}{3}$

(c) There are n kinds of fruits at the market, and you want to end up with at least two different types of fruit.

$\binom{n+k-1}{n-1}$

2 Inclusion and Exclusion

What is the total number of positive integers strictly less than 100 that are also coprime to 100?

$$99 - \underbrace{49}_{\text{numbers divisible by 2}} - \underbrace{19}_{\text{numbers divisible by 5}} + \underbrace{9}_{\text{numbers divisible by 10}} = 40$$

(note: I'm counting numbers ≤ 100 rather than strictly less than 100 just to make the numbers nicer.)

3 CS70: The Musical

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

LHS: Choose 2 directors from 2n applicants.

RHS: Let first n applicants be group A and next n applicants be group B. Edward can choose either both from group A, both from group B, or one from each.

$\binom{n}{2} + \binom{n}{2} + n^2$

(b) Edward would now like to select a crew out of n people. Use this to provide a combinatorial argument that proves the following identity: (this is called Pascal's Identity)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

LHS: Choose k out of n people.

RHS: Either pick first person and choose k-1 out of remaining n-1 people, or don't pick first person and choose k people from remaining n-1.

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

LHS: For each possible group size k, there are $\binom{n}{k}$ ways to pick the group and then k ways to choose the lead.

RHS: Pick one of the n actors to be the lead first, and then for each of the n-1 other actors, decide whether they are selected or not.

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity:

$$\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}.$$

LHS: For all possible group sizes k, there are $\binom{n}{k}$ ways to pick a group and then $\binom{k}{j}$ ways to pick the j leads.

RHS: $\binom{n}{j}$ ways to pick the j leads first, then for n-j remaining people decide whether they are selected or not.